Lecture 2
$\mathbb{R}$- and $\mathbb{C}$-Differentiability

Let $z_0 = x_0 + iy_0 = (x_0, y_0)$ be a point in $\mathbb{C}$ and f a function defined on a neighbourhood of z_0 (e.g., on an open disk $\Delta(z_0, r)$ for some $r > 0$) with values in $\mathbb{C}$. Write $f(z) = \text{Re}\, f(z) + i\text{Im}\, f(z) = u(z) + iv(z) = u(x,y) + iv(x,y)$.

Definition 2.1. The function f is called $\mathbb{R}$-*differentiable at* z_0 if there exist real numbers a, b, c, d such that

$$u(x,y) = u(x_0, y_0) + a\Delta x + b\Delta y + o(\Delta x, \Delta y),$$

$$v(x,y) = v(x_0, y_0) + c\Delta x + d\Delta y + o(\Delta x, \Delta y),$$

where $\Delta x := x - x_0$, $\Delta y := y - y_0$ and $o(\Delta x, \Delta y)$ denotes any real-valued function with the property

$$\frac{o(\Delta x, \Delta y)}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \to 0 \text{ as } (\Delta x)^2 + (\Delta y)^2 \to 0.$$

Here a, b, c, d are determined uniquely and are in fact the corresponding *first-order partial derivatives of u and v at* (x_0, y_0):

$$a = \frac{\partial u}{\partial x}(x_0, y_0), \quad b = \frac{\partial u}{\partial y}(x_0, y_0), \quad c = \frac{\partial v}{\partial x}(x_0, y_0), \quad d = \frac{\partial v}{\partial y}(x_0, y_0).$$

We will now introduce the concept of complex differentiability and compare it with that of real differentiability as defined above.

Definition 2.2. The function f is said to be $\mathbb{C}$-*differentiable at* z_0 if there exists the limit

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z},$$

that is, one can find $A \in \mathbb{C}$ such that for any $\varepsilon > 0$ there is a sufficiently small $\delta > 0$ with the property

$$\left| \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} - A \right| < \varepsilon$$

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if $0 < |\Delta z| < \delta$. In this case, the complex number A, i.e., the value of the limit, is called *the derivative of f at z_0* and is denoted by $f'(z_0)$.

The usual rules from real analysis for computing the derivatives of sums, products, quotients and compositions of $\mathbb{C}$-differentiable functions work here (check!).

Definition 2.2 is equivalent to saying that for some $A \in \mathbb{C}$ the function f is represented as

$$f(z) = f(z_0) + A\Delta z + o(\Delta z), \tag{2.1}$$

where $\Delta z := z - z_0$ and $o(\Delta z)$ denotes any complex-valued function with the property

$$\frac{o(\Delta z)}{\Delta z} \to 0 \text{ as } \Delta z \to 0$$

(note that one can write $o(\Delta z)$ instead of $o(\Delta x, \Delta y)$ in Definition 2.1). By separating the real and imaginary parts in identity (2.1), we see that it is equivalent to the following pair of real identities:

$$u(x,y) = u(x_0,y_0) + \operatorname{Re} A\,\Delta x - \operatorname{Im} A\,\Delta y + o(\Delta x, \Delta y),$$

$$v(x,y) = v(x_0,y_0) + \operatorname{Im} A\,\Delta x + \operatorname{Re} A\,\Delta y + o(\Delta x, \Delta y).$$

Comparing these identities with Definition 2.1 and taking into account that the constants a,b,c,d are chosen uniquely, we obtain:

Theorem 2.1. *The function f is $\mathbb{C}$-differentiable at $z_0 = x_0 + iy_0$ if and only if it is $\mathbb{R}$-differentiable at z_0 and the first-order partial derivatives of u and v at (x_0,y_0) satisfy the relations*

$$\frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0), \quad \frac{\partial u}{\partial y}(x_0,y_0) = -\frac{\partial v}{\partial x}(x_0,y_0). \tag{2.2}$$

Relations (2.2) are called *the Cauchy-Riemann equations* or simply *the CR-equations*. We will now rewrite them in a different form.

Definition 2.3. Let f be $\mathbb{R}$-differentiable at z_0. Set

$$\frac{\partial f}{\partial z}(z_0) := \frac{1}{2}\left(\frac{\partial f}{\partial x}(x_0,y_0) - i\frac{\partial f}{\partial y}(x_0,y_0)\right), \; \frac{\partial f}{\partial \bar z}(z_0) := \frac{1}{2}\left(\frac{\partial f}{\partial x}(x_0,y_0) + i\frac{\partial f}{\partial y}(x_0,y_0)\right),$$

where

$$\frac{\partial f}{\partial x}(x_0,y_0) := \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0), \quad \frac{\partial f}{\partial y}(x_0,y_0) := \frac{\partial u}{\partial y}(x_0,y_0) + i\frac{\partial v}{\partial y}(x_0,y_0).$$

Now, multiplying the second identity from Definition 2.1 by i, adding it to the first one and substituting

$$\Delta x = \frac{\Delta z + \overline{\Delta z}}{2}, \quad \Delta y = \frac{\Delta z - \overline{\Delta z}}{2i},$$

we obtain

$$f(z)=f(z_0)+\left(\frac{\partial u}{\partial x}(x_0,y_0)+i\frac{\partial v}{\partial x}(x_0,y_0)\right)\frac{\Delta z+\overline{\Delta z}}{2}+$$
$$\left(\frac{\partial u}{\partial y}(x_0,y_0)+i\frac{\partial v}{\partial y}(x_0,y_0)\right)\frac{\Delta z-\overline{\Delta z}}{2i}+o(\Delta z)=$$
$$f(z_0)+\frac{1}{2}\left(\frac{\partial f}{\partial x}(x_0,y_0)-i\frac{\partial f}{\partial y}(x_0,y_0)\right)\Delta z+$$
$$\frac{1}{2}\left(\frac{\partial f}{\partial x}(x_0,y_0)+i\frac{\partial f}{\partial y}(x_0,y_0)\right)\overline{\Delta z}+o(\Delta z)=$$
$$f(z_0)+\frac{\partial f}{\partial z}(z_0)\Delta z+\frac{\partial f}{\partial \bar z}(z_0)\overline{\Delta z}+o(\Delta z).$$

Comparing the above formula with (2.1), we see:

Theorem 2.2. *The function f is ℂ-differentiable at z_0 if and only if it is ℝ-differentiable at z_0 and $\frac{\partial f}{\partial \bar z}(z_0)=0$. In this case $f'(z_0)=\frac{\partial f}{\partial z}(z_0)$.*

Proof. Homework. (Hint: use, e.g., Exercise 2.4.) □

Remark 2.1. The fact that the CR-equations from Theorem 2.1 are equivalent to the condition $\frac{\partial f}{\partial \bar z}(z_0)=0$ can be also seen directly from Definition 2.3 by separating the real and imaginary parts of $\frac{\partial f}{\partial \bar z}(z_0)$. Indeed, we have

$$\frac{\partial f}{\partial \bar z}(z_0)=\frac{1}{2}\left(\frac{\partial u}{\partial x}(x_0,y_0)-\frac{\partial v}{\partial y}(x_0,y_0)\right)+\frac{i}{2}\left(\frac{\partial v}{\partial x}(x_0,y_0)+\frac{\partial u}{\partial y}(x_0,y_0)\right),$$

hence the complex identity $\frac{\partial f}{\partial \bar z}(z_0)=0$ is equivalent to the pair of real identities in (2.2).

The "formal derivatives" $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \bar z}$ can be often treated as "usual" derivatives. For example, consider a complex-valued polynomial in x, y

$$P(x,y)=\sum_{0\le \ell,m\le K} a_{\ell m}x^\ell y^m,\ a_{\ell m}\in\mathbb{C}.$$

Substituting

$$x=\frac{z+\bar z}{2},\quad y=\frac{z-\bar z}{2i},$$

we express P via z, $\bar z$:

$$P=\sum_{0\le \ell,m\le K} b_{\ell m}z^\ell \bar z^m$$

for some $b_{\ell m}\in\mathbb{C}$.

Proposition 2.1. *For all $z_0 \in \mathbb{C}$ we have*

$$\frac{\partial P}{\partial z}(z_0) = \sum_{0\le \ell, m \le K} \ell b_{\ell m} z_0^{\ell-1}\bar{z}_0^m, \quad \frac{\partial P}{\partial \bar{z}}(z_0) = \sum_{0\le \ell, m \le K} m b_{\ell m} z_0^{\ell}\bar{z}_0^{m-1}.$$

Proof. It suffices to prove the proposition for any monomial $Q := z^\ell \bar{z}^m$. Since $Q = (x+iy)^\ell (x-iy)^m$, for $z_0 = x_0 + iy_0$ we calculate

$$\frac{\partial Q}{\partial x}(x_0,y_0) = \ell(x_0+iy_0)^{\ell-1}(x_0-iy_0)^m + m(x_0+iy_0)^\ell (x_0-iy_0)^{m-1} = \ell z_0^{\ell-1}\bar{z}_0^m + m z_0^\ell \bar{z}_0^{m-1},$$

$$\frac{\partial Q}{\partial y}(x_0,y_0) = i\ell(x_0+iy_0)^{\ell-1}(x_0-iy_0)^m - im(x_0+iy_0)^\ell (x_0-iy_0)^{m-1} = i\ell z_0^{\ell-1}\bar{z}_0^m - im z_0^\ell \bar{z}_0^{m-1}$$

(check!). Hence,

$$\frac{\partial Q}{\partial z}(z_0) = \frac{1}{2}\left(\frac{\partial Q}{\partial x}(x_0,y_0) - i\frac{\partial Q}{\partial y}(x_0,y_0)\right) = \ell z_0^{\ell-1}\bar{z}_0^m,$$
$$\frac{\partial Q}{\partial \bar{z}}(z_0) = \frac{1}{2}\left(\frac{\partial Q}{\partial x}(x_0,y_0) + i\frac{\partial Q}{\partial y}(x_0,y_0)\right) = m z_0^\ell \bar{z}_0^{m-1},$$

which completes the proof. □

We will be mostly interested in the $\mathbb{C}$-differentiability of functions on open subsets of the complex plane.

Definition 2.4. Let $D \subset \mathbb{C}$ be an open subset. A function $f : D \to \mathbb{C}$ is said to be *holomorphic on D* if f is $\mathbb{C}$-differentiable at every point of D. All functions holomorphic on D form a vector space over $\mathbb{C}$, which we denote by $H(D)$. Functions in $H(\mathbb{C})$ (i.e., those holomorphic on all of $\mathbb{C}$) are called *entire*.

We have the usual facts:

(1) if $f, g \in H(D)$, then $f+g \in H(D)$ and $fg \in H(D)$;
(2) if $f, g \in H(D)$ and g does not vanish at any point of D, then $f/g \in H(D)$;
(3) if $f \in H(D)$, $g \in H(G)$, $f(D) \subset G$, then $g \circ f \in H(D)$.

The proofs are identical to those for functions differentiable on open subsets of $\mathbb{R}$ from real analysis (check!).

Usually, in what follows D will be a *domain*, i.e., an open *connected* subset of $\mathbb{C}$, where connectedness is understood as the existence, for any $z, w \in D$, of *a path in D joining z and w*, that is, of a continuous map $\gamma : [0,1] \to D$ with $\gamma(0) = z$, $\gamma(1) = w$ (in this case, we sometimes say that z is the *initial point* of γ and w is its *terminal point*). Strictly speaking, the above property is called *path-connectedness* but for open subsets of $\mathbb{R}^n$ (in fact, for open subsets of any locally path-connected topological space) it is equivalent to connectedness, i.e., to the non-existence of a non-trivial subset that is both open and closed (check!).

In what follows, we will be often interested in extending a function $f \in H(D)$ holomorphically beyond the domain D. Namely, for a domain $G \supset D$ we say that f [*holomorphically*] *extends* [*to* G], or that f *can be* [*holomorphically*] *extended* [*to* G], if there exists $F \in H(G)$ such that $F(z) = f(z)$ for all $z \in D$. In this case F is called *a holomorphic extension of* f [*to* G] and we also say that f *extends to a function holomorphic on* G, or that f *can be extended to a function holomorphic on* G.

Example 2.1. Let $f(z) := z = x + iy$. Here $u = x$, $v = y$, hence

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \equiv 1, \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \equiv 0.$$

Thus, we see that f is an entire function. It then follows that every polynomial $P(z) = a_K z^K + a_{K-1} z^{K-1} + \cdots + a_0$ in z is an entire function.

Example 2.2. Let $f(z) := e^z = e^x(\cos y + i \sin y)$. Here $u = e^x \cos y$, $v = e^x \sin y$, hence

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = e^x \cos y, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -e^x \sin y.$$

Therefore, f is an entire function. Also, for any $z_0 \in \mathbb{C}$ we have

$$f'(z_0) = \frac{\partial f}{\partial z}(z_0) = \frac{1}{2}\left(\frac{\partial f}{\partial x}(x_0, y_0) - i\frac{\partial f}{\partial y}(x_0, y_0)\right) = \frac{1}{2}(e^{z_0} - i(ie^{z_0})) = e^{z_0}$$

as expected. It then follows that *the basic trigonometric functions*

$$\cos z := \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z := \frac{e^{iz} - e^{-iz}}{2i}$$

are entire as well.

Example 2.3. Let $f(z) := \bar{z} = x - iy$. Here $u = x$, $v = -y$, hence

$$\frac{\partial u}{\partial x} \equiv 1, \quad \frac{\partial v}{\partial y} \equiv -1, \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \equiv 0.$$

Therefore, f is not $\mathbb{C}$-differentiable at any point of $\mathbb{C}$.

Example 2.4. Let $f(z) := \bar{z}^2 = x^2 - y^2 - 2ixy$. Here $u = x^2 - y^2$, $v = -2xy$, hence

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial y} = -2x, \quad \frac{\partial u}{\partial y} = -2y, \quad \frac{\partial v}{\partial x} = -2y.$$

Therefore, f is only $\mathbb{C}$-differentiable at the origin. In particular, f is not holomorphic on any domain in $\mathbb{C}$.

Notice also that as a consequence of Proposition 2.1 we have:

Corollary 2.1. *The polynomial P from Proposition 2.1 is an entire function if and only if $b_{\ell,m} = 0$ for all $m > 0$. In this case the derivative $P' = \dfrac{\partial P}{\partial z}$ is given by formal differentiation with respect to z.*

Proof. Homework. (Hint: use exercise 2.10.) □

Exercises

2.1. Let f be $\mathbb{R}$-differentiable at z_0. Prove that the Jacobian of f at z_0 is equal to

$$\left|\frac{\partial f}{\partial z}(z_0)\right|^2 - \left|\frac{\partial f}{\partial \bar{z}}(z_0)\right|^2.$$

2.2. Let f be $\mathbb{R}$-differentiable at z_0. Prove that the *limit set* of the ratio

$$g(z) := \frac{f(z) - f(z_0)}{z - z_0}, \quad z \neq z_0,$$

at z_0 is the circle centred at the point $\dfrac{\partial f}{\partial z}(z_0)$ of radius $\left|\dfrac{\partial f}{\partial \bar{z}}(z_0)\right|$. Here the limit set consists of all *limit points* of g at z_0, i.e., of all complex numbers A for which there is a sequence $\{z_n\}$ not including the point z_0, with $|z_n - z_0| \to 0$ as $n \to \infty$, such that $|g(z_n) - A| \to 0$.

2.3. Let f be continuously differentiable on $\mathbb{C}$. Suppose that f preserves distances, i.e.,

$$|f(z_1) - f(z_2)| = |z_1 - z_2| \;\forall z_1, z_2 \in \mathbb{C}.$$

Prove that either $f(z) = e^{i\alpha} z + a$, or $f(z) = e^{i\alpha}\bar{z} + a$ for some $a \in \mathbb{C}$ and $\alpha \in \mathbb{R}$. (Hint: use Exercise 2.2.)

2.4. Assume that f is defined on a neighbourhood of a point z_0 and in this neighbourhood for some $A, B \in \mathbb{C}$ the following holds:

$$f(z) = f(z_0) + A\Delta z + B\overline{\Delta z} + o(\Delta z),$$

where $\Delta z := z - z_0$. Prove that f is $\mathbb{R}$-differentiable at z_0 with $A = \dfrac{\partial f}{\partial z}(z_0)$ and $B = \dfrac{\partial f}{\partial \bar{z}}(z_0)$.

2.5. Suppose that $f = u + iv$ is defined on a neighbourhood of 0 and is continuous at 0. Assume that all first-order partial derivatives of u and v exist at 0 and satisfy the CR-equations at 0. Does it follow that f is $\mathbb{C}$-differentiable at the origin? Prove your conclusion.

2.6. For each of the following functions, find all points at which it is $\mathbb{C}$-differentiable:

$$\text{(i) } z^2|z|^4,$$
$$\text{(ii) } (\operatorname{Re} z)^4,$$
$$\text{(iii) } \sin(\operatorname{Im} z).$$

2.7. Let $f = u + iv$ be $\mathbb{C}$-differentiable at z_0, with $f'(z_0) \neq 0$, and continuously differentiable on a neighbourhood of z_0. Prove that the angle between the level sets of u and v at z_0 (that is, any of the four angles between the tangent lines to the level sets at z_0) is equal to $\pi/2$. (Hint: write the tangent lines to the level sets at z_0 via the first-order partial derivatives of u and v at z_0.)

2.8. Suppose that a function f is $\mathbb{C}$-differentiable at a point z_0. Prove that the function

$$g(z) := \overline{f(\bar{z})}$$

is $\mathbb{C}$-differentiable at the point $\bar{z}_0$ and $g'(\bar{z}_0) = \overline{f'(z_0)}$.

2.9. Suppose that a function f is $\mathbb{C}$-differentiable at a point z_0 and $f'(z_0) \neq 0$. Show that for any disk $\Delta(z_0, r)$ on which f is defined the set $f(\Delta(z_0, r))$ cannot lie in a half-plane on either side of any line passing through $f(z_0)$.

2.10. Show that for a polynomial

$$P(z, \bar{z}) = \sum_{0 \leq \ell, m \leq K} b_{\ell m} z^{\ell} \bar{z}^{m}, \; b_{\ell m} \in \mathbb{C},$$

in z, $\bar{z}$ one has $P \equiv 0$ on an open subset of $\mathbb{C}$ if and only if $b_{\ell m} = 0$ for all $\ell, m \in \{0, \ldots, K\}$. (Hint: argue by induction using Proposition 2.1.)

2.11. Suppose that for a polynomial

$$P(z, \bar{z}) = \sum_{0 \leq \ell, m \leq K} b_{\ell m} z^{\ell} \bar{z}^{m}, \; b_{\ell m} \in \mathbb{C},$$

in z, $\bar{z}$ we have $b_{\ell m} \neq 0$ for some $0 \leq \ell \leq K$ and some $0 < m \leq K$. Prove that the set of points at which P is $\mathbb{C}$-differentiable is nowhere dense in $\mathbb{C}$. (Hint: use Exercise 2.10.)

2.12. Find the derivatives of $\sin z$ and $\cos z$ at an arbitrary point of $\mathbb{C}$.

2.13. Prove that $|\cos z|$ and $|\sin z|$ are not bounded on $\mathbb{C}$.

2.14. How many zeroes does the entire function $2 + \sin z$ have in $\mathbb{C}$? Find all the zeroes.

2.15. Using the power series expansions of e^x, $\cos y$, $\sin y$, find a power series expansion with centre 0 for each of e^z, $\cos z$, $\sin z$, i.e., represent each of these functions in the form

$$\sum_{n=0}^{\infty} c_n z^n \ \forall z \in \mathbb{C},$$

where $c_n \in \mathbb{C}$, $n = 0,1,2,\ldots$. (Hint: substitute the corresponding series into the expression $e^x \cos y + ie^x \sin y$ using arithmetic operations with absolutely convergent series and the possibility of re-arranging their terms.)

2.16. Prove that the function $\dfrac{\sin z}{z}$ is holomorphic on $\mathbb{C} \setminus \{0\}$ and that it holomorphically extends to $\mathbb{C}$.

2.17. Let

$$f(z) := \begin{cases} z^2 \sin \dfrac{1}{z} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$$

Is this function $\mathbb{C}$-differentiable at 0? Prove your conclusion.

2.18. Let $f(z) = u(x) + iv(y)$ be an entire function. Prove that $f(z) = az + b$, where $a \in \mathbb{R}$, $b \in \mathbb{C}$.

2.19. Let $a,b,c \in \mathbb{C}$. Write the quadratic polynomial $P(x,y) = ax^2 + bxy + cy^2$ in x, y via z and $\bar{z}$ and find necessary and sufficient conditions on a,b,c for P to be an entire function.

2.20. Find all entire functions f such that $\operatorname{Re} f(z) = x^2 - y^2$.

2.21. Find an entire functions f such that $\operatorname{Re} f(z) = x^2 - y^2 + xy$, $f(0) = 0$.

2.22. Find an entire functions f such that $\operatorname{Re} f(z) = e^x(x \cos y - y \sin y)$, $f(0) = 0$.

2.23. Let $g : [0,1] \to \mathbb{C}$ be a continuous function. For all $z \in \mathbb{C} \setminus [0,1]$ define

$$f(z) := \frac{1}{2\pi i} \int_0^1 \frac{g(t)}{t - z}\, dt, \tag{2.3}$$

where the integral is understood by separating the real and imaginary parts of the integrand. Prove that $f \in H(\mathbb{C} \setminus [0,1])$.

2.24. Construct an example showing that the Mean Value Theorem for $\mathbb{C}$-valued functions does not hold. Namely, find a differentiable function $f : [0,1] \to \mathbb{C}$ such that $f'(t) \neq f(1) - f(0)$ for all $t \in (0,1)$.

2.25. Prove the following variant of the Mean Value Theorem for $\mathbb{C}$-valued functions: if $f : [0,1] \to \mathbb{C}$ is a continuously differentiable function, then the difference $f(1) - f(0)$ lies in the closure of the convex hull of the set $\{f'(t) : t \in [0,1]\}$. (Hint: use the Newton-Leibniz formula for $\operatorname{Re} f$ and $\operatorname{Im} f$.)

2.26. Let P be a polynomial in z of positive degree and S the smallest convex polygon containing the roots of P. Prove that all roots of P' lie in S. (Hint: use Exercise 1.7.)

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